

A Rohlin Type Theorem for Automorphisms of Certain Purely Infinite C^* -Algebras

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March 1997

Abstract

We show a noncommutative Rohlin type theorem for automorphisms of a certain class of purely infinite simple C^* -algebras. This class consists of the purely infinite unital simple C^* -algebras which are in the bootstrap category \mathcal{N} and have trivial K_1 -groups.

1 Introduction

A noncommutative Rohlin type theorem is a fundamental tool for the classification theory of actions of operator algebras. This theorem was first introduced by A. Connes for single automorphisms (i.e. actions of \mathbb{Z}) of finite von Neumann algebras [3]. Subsequently it was extended for actions of more general groups [18, 19]. Also in the framework of C^* -algebras this type of theorem was established first for the UHF algebras [1, 8, 9] and recently for some AF, AT algebras and some purely infinite simple C^* -algebras [12, 13, 14]. In particular A. Kishimoto showed the Rohlin type theorem for automorphisms of the Cuntz algebras O_n with n finite [12]. Our first motivation of this paper is to obtain a similar result for the Cuntz algebra O_∞ . When n is finite, the Rohlin property of the unital one-sided shift on the UHF algebra M_{n^∞} plays a crucial role to derive Rohlin projections from outer automorphisms of O_n . However for O_∞ , there does not seem to be a similar mechanism at work. But fortunately by the progress of the classification theory of purely infinite simple C^* -algebras due to E. Kirchberg, N.C. Phillips and M. Rørdam, the Cuntz algebras O_n , $n = 2, 3, \dots, \infty$ (or more generally the purely infinite unital simple

C^* -algebras which are in the bootstrap category \mathcal{N} and have trivial K_1 -groups) can be decomposed as the crossed products of unital AF algebras by proper (i.e. non-unital) corner endomorphisms [10, 22, 23]. Moreover these non-unital endomorphisms also have the Rohlin property like the unital one-sided shift on M_{n^∞} [23]. We shall use these endomorphisms to derive Rohlin projections.

The content of this paper is as follows. In Section 2 we review several facts about the crossed products of C^* -algebras by their endomorphisms, which are the starting point of our argument. The C^* -algebras described in this section contain all the purely infinite unital simple C^* -algebras in the bootstrap category \mathcal{N} having trivial K_1 -groups and the statements mainly come from M. Rørdam's paper [23] and the remarkable classification theory by E. Kirchberg and N.C. Phillips [10, 22]. In Section 3 we show a Rohlin type theorem for approximately inner automorphisms of the C^* -algebras described in Section 2. Our claim is that for any such automorphism whose nonzero powers are all outer, it has the Rohlin property. We will meet a technical difficulty where we have to make Rohlin projections for the automorphism almost central. To overcome this difficulty we use the Rohlin property of the endomorphism which appears in the crossed product decomposition as stated above. Finally in Section 4 we present several examples of automorphisms which have the Rohlin property. Up to conjugacy these examples include well-known automorphisms of Cuntz algebras which are found in [7, 17].

2 Crossed product decomposition

We start our argument with some definitions of key words which we use throughout this paper. For details we refer to [20, 23].

Definition 1 Let α be a (unital or non-unital) endomorphism on a unital C^* -algebra A . Then α is said to have the *Rohlin property* if for any $M \in \mathbb{N}$, finite subset F of A and $\varepsilon > 0$, there exist projections $e_0, \dots, e_{M-1}, f_0, \dots, f_M$ in A such that

$$\begin{aligned} \sum_{i=0}^{M-1} e_i + \sum_{j=0}^M f_j &= 1, \\ e_i \alpha(1) &= \alpha(1) e_i, \quad f_j \alpha(1) = \alpha(1) f_j, \\ \|e_i x - x e_i\| &< \varepsilon, \quad \|f_j x - x f_j\| < \varepsilon, \\ \|\alpha(e_i) - e_{i+1} \alpha(1)\| &< \varepsilon, \quad \|\alpha(f_j) - f_{j+1} \alpha(1)\| < \varepsilon \end{aligned}$$

for $i = 0, \dots, M-1$, $j = 0, \dots, M$ and all $x \in F$, where $e_M \equiv e_0$, $f_{M+1} \equiv f_0$.

Definition 2 An endomorphism ρ on a unital C^* -algebra B is called a *corner endomorphism* if ρ is an isomorphism from B onto $\rho(1)B\rho(1)$. A corner endomorphism ρ is called a *proper corner endomorphism* if ρ is non-unital. Let ρ be

a corner endomorphism on B . Then the crossed product $B \rtimes_{\rho} \mathbb{N}$ is defined to be the universal C^* -algebra generated by a copy of B and an isometry s which implements ρ , that is, $\rho(b) = sbs^*$ for all $b \in B$.

Let \mathcal{N} be the smallest full subcategory of the separable nuclear C^* -algebras which contains the separable Type I C^* -algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by \mathbb{R} and by \mathbb{Z} [24]. A simple unital C^* -algebra A , which has at least dimension two, is said to be *purely infinite* if for any nonzero positive element $a \in A$ there exists $x \in A$ such that $axa^* = 1$. For convenience let \mathcal{A} denote the purely infinite unital simple C^* -algebras which are in the bootstrap category \mathcal{N} and have trivial K_1 -groups. According to Theorem 3.1, Proposition 3.7, Corollary 4.6 in [23] and to Theorem 4.2.4 in [22] we have the following theorem.

Theorem 3 *For any C^* -algebra A in \mathcal{A} there exist a unital simple AF algebra B with a unique tracial state, unital finite-dimensional C^* -subalgebras $(B_N \mid N \in \mathbb{N})$ of B and a proper corner endomorphism ρ on B with the Rohlin property such that*

$$\begin{aligned} A &\cong B \rtimes_{\rho} \mathbb{N}, \\ B_N &\subseteq B_{N+1}, \quad \bigcup_{N \in \mathbb{N}} B_N \text{ is dense in } B, \\ \rho(B_N) &\subseteq B_{N+1}, \quad pB_Np \subseteq \rho(B_{N+1}) \end{aligned}$$

for all $N \in \mathbb{N}$, where $p \equiv \rho(1) \neq 1$ and that p is full in B_1 , i.e. $p \in B_1$ and the linear hull of B_1pB_1 is B_1 . Conversely every C^* -algebra arising as a crossed product algebra described above and having the trivial K_1 -group is in \mathcal{A} .

Henceforth we let A denote a C^* -algebra in \mathcal{A} and let B , (B_N) , ρ , p be as in the statement of Theorem 3. Finally in this section we state some technical lemma needed later. Since p is full in B_1 we have elements a_1, \dots, a_r in B_1 such that

$$\sum_{i=1}^r a_i p a_i^* = 1, \quad a_i p = a_i.$$

Let s be an isometry in $A \cong B \rtimes_{\rho} \mathbb{N}$ which implements ρ . Define $\sigma(x) = \sum_{i=1}^r a_i s x s^* a_i^*$ for $x \in A$, then σ has the following properties ([23, Lemma 6.3.]):

Lemma 4 (1) $\sigma \upharpoonright A \cap B_2'$ is a unital $*$ -homomorphism.

(2) $\sigma(A \cap B_{N+1}') \subseteq A \cap B_N'$ for all $N \in \mathbb{N}$.

(3) $s^j x s^{*j} = \sigma^j(x) s^j s^{*j} = s^j s^{*j} \sigma^j(x)$ for all $j \in \mathbb{N}$, and $x \in A \cap B_{j+1}'$.

3 Rohlin type theorem

In this section we state the main theorem of this paper. That is

Theorem 5 *Let A be a C^* -algebra in the class \mathcal{A} . For any approximately inner automorphism α of A the following conditions are equivalent:*

- (1) α^k is outer for any nonzero integer k .
- (2) α has the Rohlin property.

Here an automorphism of a C^* -algebra is said to be *approximately inner* if it can be approximated pointwise by inner automorphisms. It is clear that (2) implies (1). To show the converse we take several steps. Since A is in \mathcal{A} we use the notation appeared in the previous section. Suppose that (1) in Theorem 5 holds. The next three lemmas follow by the methods used in [6, 12]

Lemma 6 *Let q be a projection in $A \cap B_2'$. Then*

$$c(\alpha^k \sigma(q)) = c(\alpha^k(q))$$

for any $k \in \mathbb{Z}$, where $c(\cdot)$ denotes the central support in the enveloping von Neumann algebra A^{**} of A .

Proof. Since $\alpha^k \sigma \alpha^{-k}$ is inner, we have that

$$c(\alpha^k \sigma(q)) = c(\alpha^k \sigma \alpha^{-k} \alpha^k(q)) \leq c(\alpha^k(q)) .$$

Since $\sigma(q)p = sqs^*$ by (3) of Lemma 4 we have

$$c(\alpha^k \sigma(q)) \geq c(\alpha^k(\sigma(q)p)) = c(\alpha^k(sqs^*)) = c(\alpha^k(q)) .$$

This completes the proof. ■

Let $\text{Proj}(A)$ denote the projections of a C^* -algebra A .

Lemma 7 *Let l, m and N be nonnegative integers with $N \geq l + m + 2$ and let k be a nonzero integer. Then for any nonzero projection e in $A \cap B_N'$,*

$$\inf \{ \|q \alpha^k \sigma^l(q)\| \mid q \in \text{Proj} \sigma^m(e(A \cap B_N')e) \setminus \{0\} \} = 0 .$$

Proof. First we show the lemma when $l, m = 0$. Assume that

$$\delta \equiv \inf \{ \|q \alpha^k(q)\| \mid q \in \text{Proj}(e(A \cap B_N')e) \setminus \{0\} \} > 0 . \quad (1)$$

Let $(e_{s,t}^{(j)} \mid j = 1, \dots, J_N; s, t = 1, \dots, d_j)$ be a system of matrix units for $B_N \cong \bigoplus_{j=1}^{J_N} M_{d_j}(\mathbb{C})$ and set $p^{(j)} = \sum_{s=1}^{d_j} e_{s,s}^{(j)}$. We may assume that $e \in p^{(j)} A p^{(j)}$ for some j , which satisfies $ee_{1,1}^{(j)} \neq 0$. Then it is easily verified that the set

$$\{ q \mid q \in \text{Proj}(A) \setminus \{0\}, q \leq ee_{1,1}^{(j)} \}$$

is equal to the set

$$\{qe_{1,1}^{(j)} \mid q \in \text{Proj}(e(A \cap B_N')e) \setminus \{0\}\}.$$

Combining this with the fact that α^k is outer, we obtain

$$\inf\{\|qe_{1,1}^{(j)}a\alpha^k(qe_{1,1}^{(j)})\| \mid q \in \text{Proj}(e(A \cap B_N')e) \setminus \{0\}\} = 0. \quad (2)$$

for any $a \in A \setminus \{0\}$ by virtue of [11, Lemma 1.1]. Here we have unitaries v_1, \dots, v_{d_j} in B_N such that

$$\sum_{s=1}^{d_j} v_s e_{1,1}^{(j)} v_s^* = p^{(j)}.$$

Then for any nonzero projection q in $e(A \cap B_N')e$,

$$\begin{aligned} q\alpha^k(q) &= qp^{(j)}\alpha(qp^{(j)}) \\ &= \sum_{s,t} qv_s e_{1,1}^{(j)} v_s^* \alpha^k(qv_t e_{1,1}^{(j)} v_t^*) \\ &= \sum_{s,t} v_s qe_{1,1}^{(j)} v_s^* \alpha^k(v_t) \alpha^k(qe_{1,1}^{(j)}) \alpha^k(v_t^*). \end{aligned}$$

By the first assumption (1) this implies, for some s, t

$$\|qe_{1,1}^{(j)} v_s^* \alpha^k(v_t) \alpha^k(qe_{1,1}^{(j)})\| \geq \frac{\delta}{d_j^2}.$$

But this inequality contradicts (2). Thus we arrive at the result when $l, m = 0$. For general l and m , using the above result we obtain a projection p_1 in $e(A^{**} \cap B_N')e$ such that p_1 is minimal in A^{**} and $c(p_1)c(\alpha^k(p_1)) = 0$. Set $p_2 = \sigma^m(p_1)$, then by Lemma 6

$$\begin{aligned} c(\alpha^k \sigma^l(p_2)) &= c(\alpha^k \sigma^{l+m}(p_1)) = c(\alpha^k(p_1)), \\ c(p_2) &= c(\sigma^m(p_1)) = c(p_1). \end{aligned}$$

Approximating p_1 by projections in $e(A \cap B_N')e$ we obtain the result. ■

Lemma 8 *Let K, L and N be positive integers with $N \geq K + L + 2$ and let $\varepsilon > 0$. Then there exists a nonzero projection e in $A \cap B_N'$ such that*

$$[e] = 0 \quad \text{in } K_0(A \cap B_N')$$

$$\|\alpha^{k_1} \sigma^{l_1}(e) \cdot \alpha^{k_2} \sigma^{l_2}(e)\| < \varepsilon$$

for $k_1, k_2 = 0, \dots, K$ and $l_1, l_2 = 0, \dots, L$ with $(k_1, l_1) \neq (k_2, l_2)$.

Proof. From the Rohlin property of ρ , we have a nonzero projection p_1 in $A \cap B_N'$ such that

$$\|p_1 \sigma^l(p_1)\| < \varepsilon$$

for $l = 1, \dots, L$. Using Lemma 7 with $m = 0$, we find a nonzero projection p_2 in $p_1(A \cap B_N')p_1$ such that

$$\|p_2 \alpha^k \sigma^l(p_2)\| < \varepsilon$$

for $k = \pm 1, \dots, \pm K$ and $l = 0, \dots, L$. Again using Lemma 7 with $m = 1$, we find a nonzero projection p_3 in $p_2(A \cap B_N')p_2$ such that

$$\|\sigma(p_3) \alpha^k \sigma^{l+1}(p_3)\| < \varepsilon$$

for any k and l as above. Repeating this application of Lemma 7 until $m = L$ we obtain a nonzero projection p_{L+2} in $A \cap B_N'$ such that

$$\|\sigma^j(p_{L+2}) \alpha^k \sigma^{l+j}(p_{L+2})\| < \varepsilon$$

for $j = 0, \dots, L$, $k = \pm 1, \dots, \pm K$ and $l = 0, \dots, L$. Since A is purely infinite we can find a nonzero projection e in $p_{L+2}(A \cap B_N')p_{L+2}$ such that $[e] = 0$ in $K_0(A \cap B_N')$. This projection e satisfies the required condition. \blacksquare

Lemma 9 *Let N be a positive integer and let $\{p^{(j)} \mid j = 1, \dots, J_N\}$ be the set of minimal central projections in B_N with $\sum_{j=1}^{J_N} p^{(j)} = 1$. Then there exist positive integers $N_1 \geq N$ which satisfy the following condition:*

$$p^{(j)} \sigma^l(q) p^{(j)} \neq 0$$

for all integer $l \geq 0$, nonzero projection q in $A \cap B_{N_1+l}'$ and $j = 1, \dots, J_N$.

Proof. By the simplicity of B we choose $N_1 \leq N$ such that the central support of $p^{(j)}$ in B_{N_1} is 1 for all j . Then for any nonzero projection $q \in A \cap B_{N_1+l}'$, it follows that $\sigma^l(q) p^{(j)} \neq 0$ since $\sigma^l(q) \in A \cap B_{N_1+l}'$. This completes the proof. \blacksquare

The next lemma says that we almost find Rohlin projections if we drop the condition that the sum of the projections is 1.

Lemma 10 *Let M, N be positive integers and let $\varepsilon > 0$. Then there exist mutually orthogonal nonzero projections e_0, \dots, e_{M-1} in A such that*

$$\|\alpha(e_i) - e_{i+1}\| < \varepsilon,$$

$$e_i \in B_N', \quad \|e_i s - s e_i\| < \varepsilon$$

for $i = 0, \dots, M-1$, where $e_M = e_0$.

Proof. Let N_0 be a positive integer which we shall make very large later. Take the minimal central projections $\{p^{(j)} | j = 1, \dots, J_{2N_0}\}$ in B_{2N_0} such that $\sum_{j=1}^{J_{2N_0}} p^{(j)} = 1$. Using Lemma 9 with $N = 2N_0$, we find positive integers $N_1 \geq 2N_0$ which satisfies the condition in Lemma 9. Let N_2, m be positive integers with $N_2 \gg N_0, N_1$ and let $\varepsilon_2 > 0$. From Lemma 8 we obtain mutually orthogonal nonzero projections $E(k, l)$ ($k = 0, \dots, mM - 1$, $l = 0, \dots, N_0 - 1$) in A such that

$$\begin{aligned} \|E(k, l) - \alpha^k \sigma^l(E(0, 0))\| &< \varepsilon_2, \\ E(0, l) &\in B_{N_2}' , \\ [E(0, l)] &= 0 \text{ in } K_0(A \cap B_{N_2}') \end{aligned}$$

for any k, l . Furthermore by the property of N_1 , we may assume that

$$p^{(j)} E(0, 0) p^{(j)}, p^{(j)} E(0, 1) p^{(j)} \neq 0$$

for each j . Thus noting that $[E(0, 0)] = [E(0, 1)] = 0$ in $K_0(A \cap B_{2N_0}')$, we have a partial isometry w_1 in $A \cap B_{2N_0}'$ such that

$$w_1^* w_1 = E(0, 0), \quad w_1 w_1^* = E(0, 1).$$

Since $\|\sigma(E(0, l)) - E(0, l+1)\| < 2\varepsilon_2$, if ε_2 is sufficiently small then we can find a unitary u_1 in $A \cap B_{2N_0}'$ such that

$$\|u_1 - 1\| < 10N_0\varepsilon_2,$$

$$Ad u_1 \circ \sigma(E(0, l)) = E(0, l+1)$$

for $l = 0, 1, \dots, N_0 - 2$. Indeed u_1 is taken as follows. Set

$$\begin{aligned} x &= \sum_{l=0}^{N_0-2} E(0, l+1) \sigma(E(0, l)) \\ &\quad + (1 - \sum_{l=1}^{N_0-1} E(0, l)) (1 - \sum_{l=0}^{N_0-2} \sigma(E(0, l))) . \end{aligned}$$

Then $x \in A \cap B_{N_2-1}'$ and

$$\begin{aligned} x - 1 &= \sum_{l=0}^{N_0-2} E(0, l+1) \{\sigma(E(0, l)) - E(0, l+1)\} \\ &\quad + (1 - \sum_{l=1}^{N_0-1} E(0, l)) (- \sum_{l=0}^{N_0-2} \sigma(E(0, l))) . \end{aligned}$$

Thus

$$\|x - 1\| \leq 2\varepsilon_2 + 2\varepsilon_2(N_0 - 1) = 2N_0\varepsilon_2,$$

$$\|xx^* - 1\| \leq 6N_0\varepsilon_2 .$$

So let $u_1 = (xx^*)^{-\frac{1}{2}}x$ then u_1 is a desired unitary in $A \cap B_{N_2-1}'$. Let $\sigma_1 = \text{Ad } u_1 \circ \sigma$ and define

$$E_{i,j} = \begin{cases} \sigma_1^{i-1}(w_1)\sigma_1^{i-2}(w_1)\cdots\sigma_1^j(w_1) & (i > j) \\ E_{(0,i)} & (i = j) \\ \sigma_1^i(w_1)^*\sigma_1^{i+1}(w_1)^*\cdots\sigma_1^{j-1}(w_1)^* & (i < j) . \end{cases}$$

Then we can easily verify that $(E_{i,j} \mid i, j = 0, \dots, N_0 - 1)$ forms a system of matrix units in $A \cap B_{N_0+2}'$. Furthermore define

$$E_0 = \frac{1}{N_0} \sum_{i,j=0}^{N_0-1} E_{i,j} .$$

Then E_0 is a nonzero projection in $A \cap B_{N_0+2}'$. Noting that $\sigma_1(E_{i,j}) = E_{i+1,j+1}$ we have

$$\begin{aligned} \|\sigma_1(E_0) - E_0\| &= \left\| \frac{1}{N_0} \sum_{i=0}^{N_0-1} \sigma_1(E_{N_0-1,i}) + \frac{1}{N_0} \sum_{i=0}^{N_0-2} \sigma_1(E_{i,N_0-1}) \right. \\ &\quad \left. - \frac{1}{N_0} \sum_{i=0}^{N_0-1} \sigma_1(E_{0,i}) - \frac{1}{N_0} \sum_{i=1}^{N_0-1} \sigma_1(E_{i,0}) \right\| \\ &\leq \frac{4}{\sqrt{N_0}} . \end{aligned}$$

Thus

$$\begin{aligned} \|\sigma(E_0) - E_0\| &\leq \|\sigma_1(E_0) - E_0\| + 2\|u_1 - 1\| \\ &\leq \frac{4}{\sqrt{N_0}} + 20N_0\varepsilon_2 . \end{aligned}$$

Accordingly

$$\begin{aligned} \|sE_0 - E_0s\| &\leq \|sE_0s^* - E_0ss^*\| = \|(\sigma(E_0) - E_0)p\| \\ &\leq \frac{4}{\sqrt{N_0}} + 20N_0\varepsilon_2 . \end{aligned}$$

Therefore if we make N_0 sufficiently large and ε_2 sufficiently small with $\varepsilon_2 \ll N_0^{-1}$, then E_0 becomes almost central in A . Consequently, for any positive integer N_3 and $\varepsilon_3 > 0$, we obtain mutually orthogonal projections E_0, \dots, E_{mM-1} in $A \cap B_{N_3}'$ such that

$$\begin{aligned} \|E_k s - s E_k\| &< \varepsilon_3 , \\ \|\alpha^k(E_0) - E_k\| &< \varepsilon_3 \end{aligned} \tag{3}$$

for $k = 0, \dots, mM - 1$. In particular taking N_3 sufficiently large and ε_3 sufficiently small, we can make $\sigma(E_k) = \sum_{i=1}^r a_i s E_k s^* a_i^*$ very close to E_k . Thus we have partial isometries u_2, v_2 in $A \cap B_{N_3-1}'$ such that

$$Ad u_2(\sigma(E_0)) = E_0 ,$$

$$\|u_2 - E_0\| \leq 2\|\sigma(E_0) - E_0\| \leq 2\left(\sum_{i=1}^r \|a_i\|\right)\varepsilon_3 ,$$

$$Ad v_2(\sigma(E_1)) = E_1 ,$$

$$\|v_2 - E_1\| \leq 2\|\sigma(E_1) - E_1\| \leq 2\left(\sum_{i=1}^r \|a_i\|\right)\varepsilon_3 .$$

Furthermore by Lemma 11 below we may assume that there is a partial isometry w_2 in $A \cap B_{N_3-1}'$ such that $w_2^* w_2 = E_0$ and $w_2 w_2^* = E_1$. Set $w_3 = v_2 \sigma(w_2) u_2^*$ then $w_3^* w_2$ is a unitary in $E_0(A \cap B_{N_3-1}')E_0$. Since $K_1(A) = 0$, $w_3^* w_2$ is homotopic to E_0 in $E_0(A \cap B_{N_3-1}')E_0$ from [15, Lemma 2.3.] and [23, Lemma 6.6.]. By virtue of the stability of $Ad u_2 \circ \sigma$ ([23, Lemma 6.4., 6.5.]), if we make N_3 sufficiently large and ε_3 sufficiently small for any positive integer N_4 and $\varepsilon_4 > 0$, we obtain a unitary y in $E_0(A \cap B_{N_4}')E_0$ such that

$$\|w_3^* w_2 - (Ad u_2 \circ \sigma)(y) y^*\| < \varepsilon_4 .$$

Thus

$$\|w_2 y - v_2 \sigma(w_2 y) u_2^*\| < \varepsilon_4 .$$

Set $W = w_2 y$ then W is a partial isometry in $A \cap B_{N_4}'$ from E_0 onto E_1 which satisfies that

$$\begin{aligned} \|W - \sigma(W)\| &\leq \|W - v_2 \sigma(W) u_2^*\| + \|(v_2 - E_1) \sigma(W) u_2^*\| \\ &\quad + \|E_1 \sigma(W) (u_2 - E_0)^*\| + \|E_1 \sigma(W) E_0 - \sigma(E_1) \sigma(W) \sigma(E_0)\| \\ &\leq \varepsilon_4 + 4\left(\sum_{i=1}^r \|a_i\|\right)\varepsilon_3 + 2\left(\sum_{i=1}^r \|a_i\|\right)\varepsilon_3 \leq 2\varepsilon_4 . \end{aligned}$$

Accordingly we have

$$\|sW - Ws\| = \|sW s^* - Ws s^*\| = \|(\sigma(W) - W)p\| \leq 2\varepsilon_4 ,$$

$$\|s^* W - W s^*\| = \|s s^* W - s W s^*\| = \|p(\sigma(W) - W)\| \leq 2\varepsilon_4 .$$

Therefore W is almost central in A when N_4 is very large and ε_4 is very small. On the other hand, by (3) we have a unitary u_3 in A such that

$$\|u_3 - 1\| < 10mM\varepsilon_3 ,$$

$$Ad u_3 \circ \alpha(E_k) = E_{k+1}$$

for $k = 0, \dots, mM - 1$. Let $\alpha_1 = \text{Ad } u_3 \circ \alpha$ and define

$$f_{i,j} = \begin{cases} \alpha_1^{i-1}(W)\alpha_1^{i-2}(W)\cdots\alpha_1^j(W) & (i > j) \\ E_i & (i = j) \\ \alpha_1^i(W)^*\alpha_1^{i+1}(W)^*\cdots\alpha_1^{j-1}(W)^* & (i < j) \end{cases}.$$

Then $(f_{i,j} \mid i, j = 0, \dots, mM - 1)$ forms a system of matrix units. Furthermore define

$$F_i = \frac{1}{m} \sum_{k,l=0}^{m-1} f_{i+kM, i+lM}$$

for $i = 0, \dots, M - 1$. It is easy to verify that F_0, \dots, F_{M-1} are mutually orthogonal and satisfy that

$$\alpha_1(F_i) = F_{i+1}, \quad \|\alpha_1(F_{M-1}) - F_0\| < \frac{4}{\sqrt{m}}$$

for $i = 0, \dots, M - 1$. Hence if we make N_4, m sufficiently large and ε_4 sufficiently small, we can obtain projections $(e_i \mid i = 0, \dots, M - 1)$ which satisfy the required condition except that $e_i \in B_N'$. But e_i 's are almost central, therefore by [2, Theorem 5.3] we have desired projections after a small inner perturbation. ■

Lemma 11 *Let α be an approximately inner automorphism of a unital purely infinite C^* -algebra A . If $(p_j \mid j \in \mathbb{N})$ is a uniformly central sequence of projections in A then for any $\varepsilon > 0$ and any unital finite dimensional C^* -subalgebra F of A there exist a $j \in \mathbb{N}$ and a partial isometry w in A such that*

$$w^*w = p_j, \quad \|ww^* - \alpha(p_j)\| < \varepsilon,$$

$$\|wx - xw\| \leq \varepsilon\|x\|$$

for any $x \in F$.

Proof. Since α is approximately inner and F is finite dimensional, the restriction of α to F is inner i.e. there exists a unitary u in A such that $\alpha \upharpoonright F = \text{Ad } u \upharpoonright F$. From uniform centrality of (p_j) , we find a sufficiently large $j \in \mathbb{N}$ such that

$$\|\alpha(p_j)u - u\alpha(p_j)\| < \varepsilon,$$

$$\|p_jx - xp_j\| \leq \varepsilon\|x\|$$

for any $x \in F$. For these u and p_j , since α is approximate inner, we have a unitary v in A such that

$$\|(\text{Ad } u^* \circ \alpha - \text{Ad } v)x\| \leq \varepsilon\|x\|$$

for any $x \in F \cup \{p_j\}$. Since $\text{Ad } u^* \circ \alpha \upharpoonright F$ is the identity, it follows that

$$\|x - vxv^*\| \leq \varepsilon\|x\|$$

for any $x \in F$. If we set $w = vp_j$, we have $w^*w = p_j$, $\|wx - xw\| \leq 2\varepsilon\|x\|$ for any $x \in F$ and

$$\begin{aligned}\|ww^* - \alpha(p_j)\| &= \|(Ad v - Ad u^* \circ \alpha)(p_j)\| + \|(Ad u^* \circ \alpha - \alpha)(p_j)\| \\ &\leq \varepsilon + \varepsilon.\end{aligned}$$

This completes the proof. ■

Proof of Theorem 5.

We have already shown in Lemma 10 that we almost have Rohlin projections except that the sum of the projections is 1. To derive genuine Rohlin projections, we can exactly follow the method of Proof of Theorem 3.1 in [12], replacing almost Φ -invariance there by almost commutativity with $B_N \cup \{s, s^*\}$ as in Lemma 10. In this process the number of towers of projections increases from one to two as in Definition 1. We have thus proved the theorem. ■

4 Examples

In this section we present several examples of automorphisms which have the Rohlin property. Let A be a C^* -algebra in \mathcal{A} and let $B \rtimes_{\rho} \mathbb{N}$ be a crossed product decomposition of A as in Section 2. By the universality of the crossed product we have the dual action $\hat{\rho}$ of \mathbb{T} on A , that is, we define $\hat{\rho}$ by the formulas: $\hat{\rho}(b) = b$, $\hat{\rho}_{\lambda}(s) = \lambda s$ for all $b \in B$, $\lambda \in \mathbb{T}$. Using the universality similarly for an automorphism α of B with $\alpha \circ \rho = \rho \circ \alpha$, we define an automorphism $\tilde{\alpha}$ of $B \rtimes_{\rho} \mathbb{N}$ by $\tilde{\alpha}(b) = \alpha(b)$ for all $b \in B$ and by $\tilde{\alpha}(s) = s$. Clearly $\tilde{\alpha}$ commutes with each $\hat{\rho}_{\lambda}$ from the definition. Then we have

Proposition 12 *An automorphism $\tilde{\alpha} \circ \hat{\rho}_{\lambda}$ of $A \cong B \rtimes_{\rho} \mathbb{N}$ is approximately inner for any $\lambda \in \mathbb{T}$, and one has the following:*

- (1) *If α is the identity mapping on A then $\tilde{\alpha} \circ \hat{\rho}_{\lambda} = \hat{\rho}_{\lambda}$ is outer for any $\lambda \in \mathbb{T} \setminus \{0\}$.*
- (2) *If α is outer (as an automorphism of B) then $\tilde{\alpha} \circ \hat{\rho}_{\lambda}$ is outer for any $\lambda \in \mathbb{T}$.*
- (3) *If α is inner then $\tilde{\alpha} \circ \hat{\rho}_{\lambda}$ are inner for at most a countable number of $\lambda \in \mathbb{T}$.*

Therefore in any case $\tilde{\alpha} \circ \hat{\rho}_{\lambda}$ have the Rohlin property for an uncountable number of $\lambda \in \mathbb{T}$.

Proof. The dual action $\hat{\rho}$ of \mathbb{T} on $B \rtimes_{\rho} \mathbb{N}$ is strongly continuous. Hence $\hat{\rho}_{\lambda}$ is approximately inner by virtue of Rørdam's classification theorem [23, Theorem 6.12.]. Since A is isomorphic to a corner of $(B \otimes \mathbb{K}) \rtimes_{\beta} \mathbb{Z}$ for some automorphism

β of $B \otimes \mathbb{K}$ by [20], it follows that $A \rtimes_{\hat{\rho}} \mathbb{T}$ is a corner of $B \otimes \mathbb{K}$ by Takai's duality theorem. In particular $A \rtimes_{\hat{\rho}} \mathbb{T}$ is prime. Combining this fact with the simplicity of A , we have that $\hat{\rho}_\lambda$ is outer for any $\lambda \in \mathbb{T} \setminus \{1\}$ from Theorem 8.10.10 and Theorem 8.11.10 in [21]. This has shown (1). To show (2) assume that $\tilde{\alpha} \circ \hat{\rho}_\lambda$ is inner, that is, there is a unitary v in $B \rtimes_{\rho} \mathbb{N}$ such that $\tilde{\alpha} \circ \hat{\rho}_\lambda = Ad v$. Then since $\tilde{\alpha}$ commutes with $\hat{\rho}_\lambda$ we have that $(\tilde{\alpha} \circ \hat{\rho}_\lambda) \circ \hat{\rho}_\mu = \hat{\rho}_\mu \circ (\tilde{\alpha} \circ \hat{\rho}_\lambda)$, and that

$$Ad v \circ \hat{\rho}_\mu = \hat{\rho}_\mu \circ Ad v = Ad \hat{\rho}_\mu(v) \circ \hat{\rho}_\mu$$

for any $\mu \in \mathbb{T}$. Since $B \rtimes_{\rho} \mathbb{N}$ is simple it follows that there exists a scalar c_μ in \mathbb{T} with $v^* \hat{\rho}_\mu(v) = c_\mu 1$. It is easily checked that $c_{\mu\nu} = c_\mu c_\nu$ for all $\mu, \nu \in \mathbb{T}$, thus there is an integer k satisfying $c_\mu = \mu^k$ for all $\mu \in \mathbb{T}$. Accordingly

$$\hat{\rho}_\mu(v s^{*k}) = \mu^k v \overline{\mu}^k s^{*k} = v s^{*k}$$

for any $\mu \in \mathbb{T}$, hence $v s^{*k} \in B$. This implies that $k = 0$ because B has no proper coisometry. Therefore $v \in B$ and $\alpha = Ad v$ is inner. This proves (2). Finally we show (3). Suppose that $\lambda_1, \lambda_2 \in \mathbb{T}$ with $\lambda_1 \neq \lambda_2$ and that $\tilde{\alpha} \circ \hat{\rho}_{\lambda_1}$, $\tilde{\alpha} \circ \hat{\rho}_{\lambda_2}$ is inner. There are unitaries v_1, v_2 in $B \rtimes_{\rho} \mathbb{N}$ such that $\tilde{\alpha} \circ \hat{\rho}_{\lambda_i} = Ad v_i$, $i = 1, 2$. Then $\lambda_i s = v_i s v_i^*$ and it follows that

$$s^k v_i s^{*k} = \overline{\lambda_i}^k s^k s^{*k} v_i s^k s^{*k}$$

for all $k \in \mathbb{N}$. Thus

$$\begin{aligned} \|v_1 - v_2\| &\geq \|s^k s^{*k} (v_1 - v_2) s^k s^{*k}\| = \|\lambda_1^k s^k v_1 s^{*k} - \lambda_2^k s^k v_2 s^{*k}\| \\ &= \|\lambda_1^k v_1 - \lambda_2^k v_2\| = \|(\lambda_1 \overline{\lambda_2})^k v_1 v_2^* - 1\|. \end{aligned}$$

Here it is obvious that $\|(\lambda_1 \overline{\lambda_2})^k v_1 v_2^* - 1\| \geq 1$ for some k , so $\|v_1 - v_2\| \geq 1$. Therefore $\tilde{\alpha} \circ \hat{\rho}_\lambda$ are inner for at most a countable number of $\lambda \in \mathbb{T}$ since B is separable. We have shown (3), thereby completing the proof. \blacksquare

Acknowledgments

The author would like to express his gratitude to Professor Akitaka Kishimoto for suggesting this line of the research and for helpful discussions.

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